On the Physical Parameters Affecting Musical Qualities:

Bowed Strings of Musical Instruments

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Abstract

The goal of this project was to create a model of a bowed string so that input of the physical parameters of a string would produce the characteristic sound of a bowed string with such parameters. Such a model was not found, but important information is discussed and methods for future study is suggested.

Introduction

In the area of solving for musical instruments and musical systems there has been much work done on plucked strings, plucked string instruments and bowed string instruments, but little work since Helmholtz on the bowed string. In considering the importance of the motion of a bowed string, consider how currently electric violins, violas, etc., and bridge mounted pick-ups do not depend on the body of an instrument in the sound production, yet the sound is distinctly that of a bowed instrument. Much emphasis is put on the body shape, material properties, varnish, etc., of the instrument, but in the end the instrument is only an amplifier for the vibrations of the string.

The qualities of a string have a very strong musical importance. There are many different types and brands of commercial strings available. The physical properties of strings varies greatly, and a trained musician can easily tell the difference in sound. Through trial and error, a musician can find which types of strings he or she prefers. Over many years (steel strings were first introduced almost 200 years ago) the design of commercial strings has been optimized. With the use of a computerized model it would be easier to pick the right string for a particular musician and instrument, understand how the current design and properties of commercial strings optimizes the musical qualities, and design new types of strings.

Theory

This project is an attempt to use knowledge and theory that has been discovered over the last one hundred years and a new set of empirical data to solve for a bowed string.

The frequency of vibration of a string is given by:

Eq. 1: $$f_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}}$$

In Eq. 1, n is any positive integer. [n=1] gives the fundamental frequency while higher values give the harmonics. For this project units of cenitmeter-gram-second are used. L is the length of the string given in centimeters. T is the tension of the string in dynes. μ is the linear density given in grams per centimeter. This equation is useful for determining an unknown value when the other values are known, but does not help in determining the relative amplitudes of the harmonics. The general form for the displacement of a vibrating string is given by:

$$y(x, t) = \sum_{n=1}^{\infty} a_n \mathrm{Sin}\left[\frac{n\pi x}{L}\right] \mathrm{Sin}[\omega_n t]$$

Eq. 2:

We are not able to measure the displacement of all points along the string, so the spatial part of the expression is not of any use. There is one location on the string that is measurable, and that is at the bridge. The expression for the displacement of the bridge is:

Eq. 3:
$$y(t) = \sum_{n=1}^{\infty} a_n \, \mathrm{Sin}[\omega_n \, t]$$

For a perfectly flexible string, $\omega_n = n\omega = 2\pi * nf_0$, but for strings with finite stiffness the higher modes of vibration are not perfectly harmonic. They are shifted to higher frequencies.

The modes of vibration can be given as:

Eq. 4:
$$f_n \approx \frac{n}{2L}\sqrt{\frac{T}{\mu}}\left(1+\frac{2}{L}\sqrt{\frac{QS\kappa^2}{T}}\right)$$

Eq. 4 is found in Morse and Ingard 1968. Q is Young's Modulus. S is the area of the cross section of the string. κ is the radius of gyration. (κ=r/2 for a circular cross section.) If we let $\delta = \sqrt{QS\kappa^2}$ and call δ the stiffness factor, then we have:

Eq. 5:
$$f_n = nf_0\left(1+\frac{2\delta}{L\sqrt{T}}\right)$$

It is now clear how the stiffness, length and tension affect the harmonic shifting. All of the above values can be measured independently, but for a commercial string it would be very difficult to measure Q (Young's Modulus) because the strings are of inhomogeneous cross-section, and the Q found by bending the string over a large angle might be different than the characteristic Q in small angles of deflection, so it would be best to find Q or δ by fitting the above functions to empirical data.

Finding a function to describe a_n is much more involved. The factors that determine a_n include but are not limited to: driving function of the bow, the impedance of the end support and internal damping.

First we look at the driving function of the bow. Helmholtz discovered in 1877 that the motion of the string is not as simple as it appears to the eye, but the slip stick action of the bow on the string creates a kink that travels along the envelope of the string. It is this parabolic envelope that is observed by the eye. The complications of the bow-string interaction can be simplified by simply considering the resulting motion of the string. The motion of the kink can be described by the following parametric equations written in Mathematica:

Eq. 6:
$$x = long * \mathrm{FractionalPart}\left[\frac{c*t}{long}\right] * \mathrm{Sign}\left[\mathrm{Sin}\left[\frac{\pi*c*t}{long}\right]\right] + long * \mathrm{UnitStep}\left[-\mathrm{Sin}\left[\frac{\pi*c*t}{long}\right]\right]$$
$$y = amplitude * \mathrm{Sin}\left[\frac{c*\pi*t}{long}\right]$$

To clarify, the pair of parametric equations in Eq. 6 do not describe the motion of the string, they describe the motion of the kink. The shape of the string at any moment is two straight lines from the ends to the kink. This is the resulting plot from the above parametric equations:

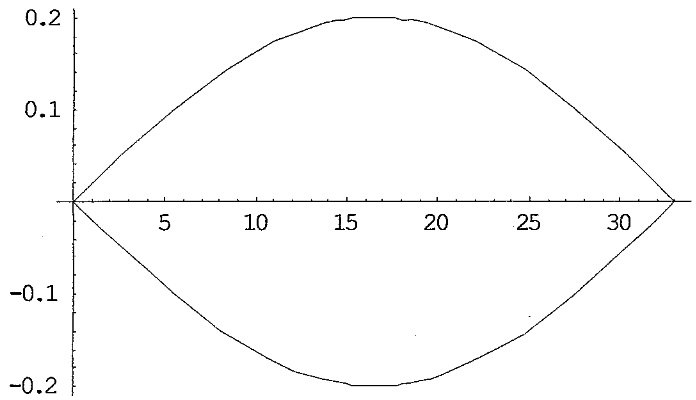

Figure 1: Spatial Plot of "Kink" in String

The transverse force on the bridge is the tension of the string times the slope at the bridge, which is simply y over x because the string makes a straight line. With a correction for the negative slope, the derivative is written as:

Eq. 7: $$\frac{y}{x} * \text{Sign}\left[\text{Sin}\left[\frac{\pi * c * t}{2 * \text{long}}\right]\right]$$

The temporal plot of Eq. 7 is shown here:

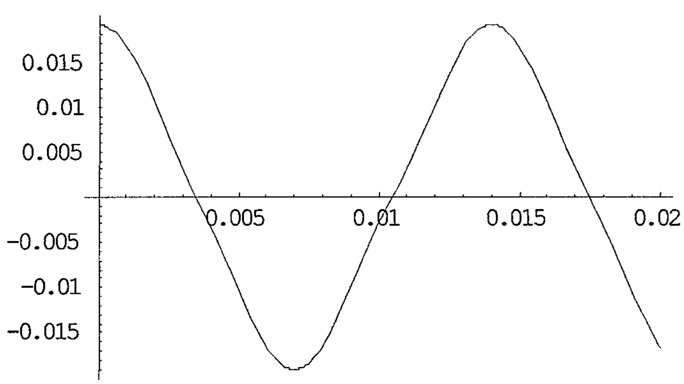

Figure 2: Temporal Plot of Transverse Force on Bridge

This is similar to a triangle wave. The function was converted to a data file, then fit to a trig function so that the relative amplitudes of the harmonics could be determined.

Figure 3 is a graphical representation of the harmonic amplitudes shown with the power spectrum of a triangle wave for reference.

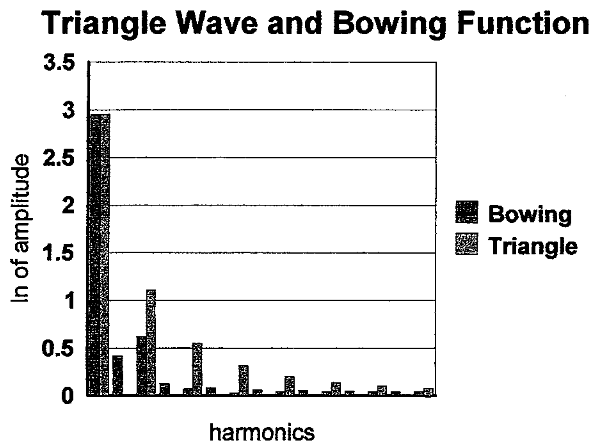

Figure 3: Relative Amplitudes of Bowing Function and Triangle Wave

The amplitude levels shown in Figure 3 are the basis for determining a_n . The other factors such as internal damping and bridge impedance will raise or lower these amplitude levels according to the theory presented next. Note that Figure 3 shows the natural logs of the amplitudes. Also notice that the points for the triangular wave form a smooth curve, but the points for the bow function do not follow any discernible pattern, and so no attempt is made to try to fit this data to a function.

The next factor that affects a_n is the internal damping. An example of how internal damping affects sound is that of a guitar with steel strings versus nylon strings. Steel strings give a brilliant sound and nylon strings give a mellow sound. The nylon strings have higher internal damping. Internal damping is frequency dependent and damps out the higher harmonics. The difference in tone quality is due to the higher internal damping of nylon. A good description of internal damping is found in Fletcher and Rossing 1991. Internal damping can be understood by considering Young's Modulus to be $Q = Q_1 + iQ_2$. Q_1 creates an instantaneous strain to an applied stress, and Q_2 creates a strain after a characteristic time τ . The energy lost to internal damping is proportional to $\pi f Q_2/Q_1$. Q_2/Q_1 may be less than 10^{-4} in some hard crystals and as high as 10^{-1} in some polymer materials (Fletcher and Rossing 1991). Note how internal damping is frequency dependent. I do not have the means to measure Q_2/Q_1. I had planned on finding published values of Q_2/Q_1 for certain materials such as nylon and steel, finding homogenous wires of those materials, testing those wires, and interpolating the data to find Q_2/Q_1 of the strings. Unfortunately, after an extensive search I was not able to find any other information on the complex Young's Modulus.

The final factor that affects a_n is the energy lost to the end support or bridge. The first thing to consider in finding the energy transfer to the bridge is the impedance (Z). Impedance is the ratio of force to velocity. Since the impedance of the bridge is much greater than that of the string (i.e. the bridge moves very little compared to the string), the energy transfer can be approximated as Z_{string} / Z_{bridge}. The impedance of a string is $\sqrt{T\mu}$. This theory matches observation as instruments with more tension on the strings produce a louder sound. The effect of linear density is not so obvious because as linear density increases, the pitch is lowered, and the perceived sound intensity is less for lower tones.

For a better understanding of impedance, consider the bridge as a driven oscillator.

Eq. 8: $$\frac{d^2 x}{dt^2} + 2\beta \frac{dx}{dt} + \omega_0^2 x = a e^{-2\pi i f t}$$

where: $a = \frac{F}{m}$ $\quad \beta = \frac{R}{2m}$ $\quad \omega_0^2 = \frac{K}{m}$ $\quad$ f = frequency of string

Eq. 8 is a differential equation that can be rewritten as the algebraic equation

Eq. 9: $$\left(-m\omega^2 - iR\omega + K\right) De^{-i\omega t} = Fe^{-i\omega t}$$

D is the displacement and can also be written as

Eq. 10: $$D = \frac{F}{-i\omega Z_m}$$

From Morse and Ingard

Eq. 11: $F(t)=\sqrt{(\mu T)}\, y'(t)$

The y' term in Eq. 11 should be apparent because the amplitude of vibrations on the string should be proportional to the force on the bridge. The $\sqrt{\mu T}$ term (also the impedance of the string) is not apparent and leads to a useful result. So the displacement of the bridge becomes:

Eq. 12: $$D = \frac{Z_{string}\, y'(t)}{-i\omega Z_{bridge}}$$

The mechanical impedance is given by

Eq. 13: $$Z_m = -i\omega m + R_m + i\frac{K}{\omega}$$

Since the bridge is very light and stiff we know that $m \ll K$ and that $\omega \ll \omega_0$.
Z_m can be approximated as $R_m + iK/\omega$. R_m is small, but still appreciable because it is the factor that ultimately leads to the sound production by the instrument. By fitting D to the data it will be possible to determine the relative magnitude of K and R_m.

Method

Due to the great amount of time necessary to measure and analyze data from each string, the sample strings were limited to one set of Super Sensitive Red Label violin strings. This type of string is very common.

The mass of the strings was too small to get an accurate measurement of the linear density, so the linear density was made a dependent variable with tension and length the independent variable and frequency the measured quantity.

To reduce resonances from an instrument and to allow tension to be an independent variable, measurements were not taken on an instrument, but rather an experimental set up shown here.

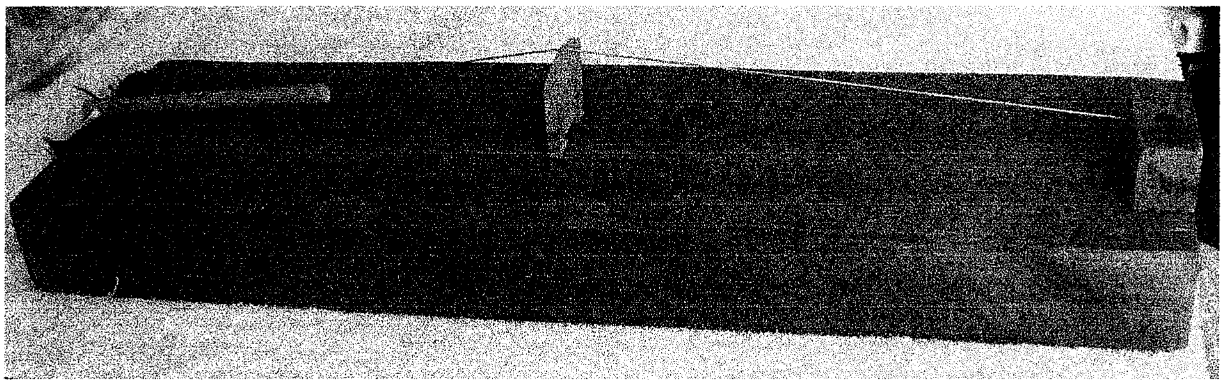

Figure 4: Experimental Apparatus

Four different strings (thus four different linear densities, stiffnesses and internal dampings) were measured with three different tensions and two different lengths for a total of 24 total data sets. Tension was applied by attaching weights to the string and hung off the side of the work bench.

Strings were bowed with the attempt to get the best sound quality possible. There are many different types of bowing styles, but using any type of bowing other than steady, sustained bowing would complicate the experiment far too much.

Three different measuring devices were available: a Shure microphone, a Super Sensitive body mounted pick up, and a Matrix bridge mounted tuning pick up. The Matrix pick up proved to be the best, giving high amplitude levels and low harmonic distortion. A test with the Super Sensitive pick up on a violin showed no noticeable difference in the sound quality when the Matrix pick up was in place. The Matrix pick up was connected to an Avance AC97 sound card. Sound samples were recorded and filtered with Magix Music Studio.

Sound samples were analyzed with Mathematica. Mathematica converts the sound files in Wave format to plain data using the command ReadSoundfile. It is then necessary to manually cut one period of the sound sample. Using the command TrigFit and a default of 15 harmonics, the data were fit to a sum of sines and cosines.

Here is an example of one such fit:

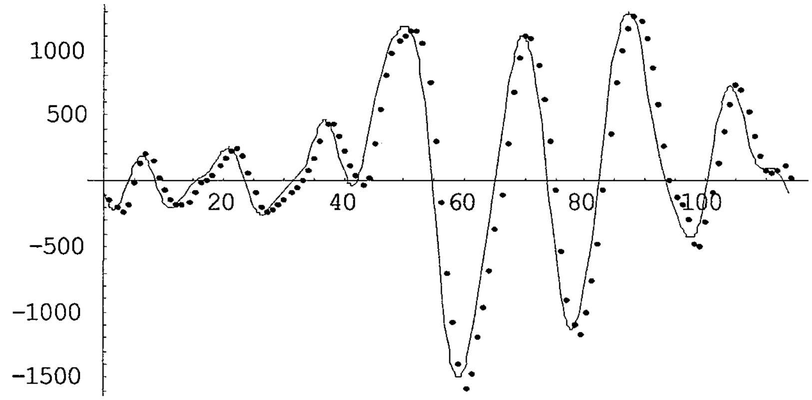

Figure 5: Fitting a Sine Function to Data Points

As you can see the fit matches well. Fifteen harmonics was enough for most data sets, but some had to be increased to as much as eighteen. I was not able to find a way to fit the data with the shifted harmonics, so all fits assume a perfectly flexible string. Even without the shifted harmonics, the data fit extremely well. This indicates that stiffness was small for these strings.

The maximum amplitude of each sound sample was noted to help determine Z_m.

Analysis

The first thing that became apparent was that the power going to the bridge was not just proportional to Z_{string}/Z_{bridge} because the impedance of the string affected how forcefully the string could be bowed. The power going to the bridge was more likely proportional to Z^2_{string}/Z_{bridge}..

The sum of sines and cosines is dependent on the relative phases of the different harmonics. The phases of harmonics are important in testing the fitting, but not important in producing sound. Our ears are phase insensitive, and adjusting the harmonics to have the same phase will make it easier to evaluate the sound.

Considering that $A\sin\theta + B\cos\phi = C\sin(\theta+\delta)$ where $C=\sqrt{A^2+B^2}$, I converted the sums of sines and cosines to a sum of sines. A sample of the coefficients are plotted here.

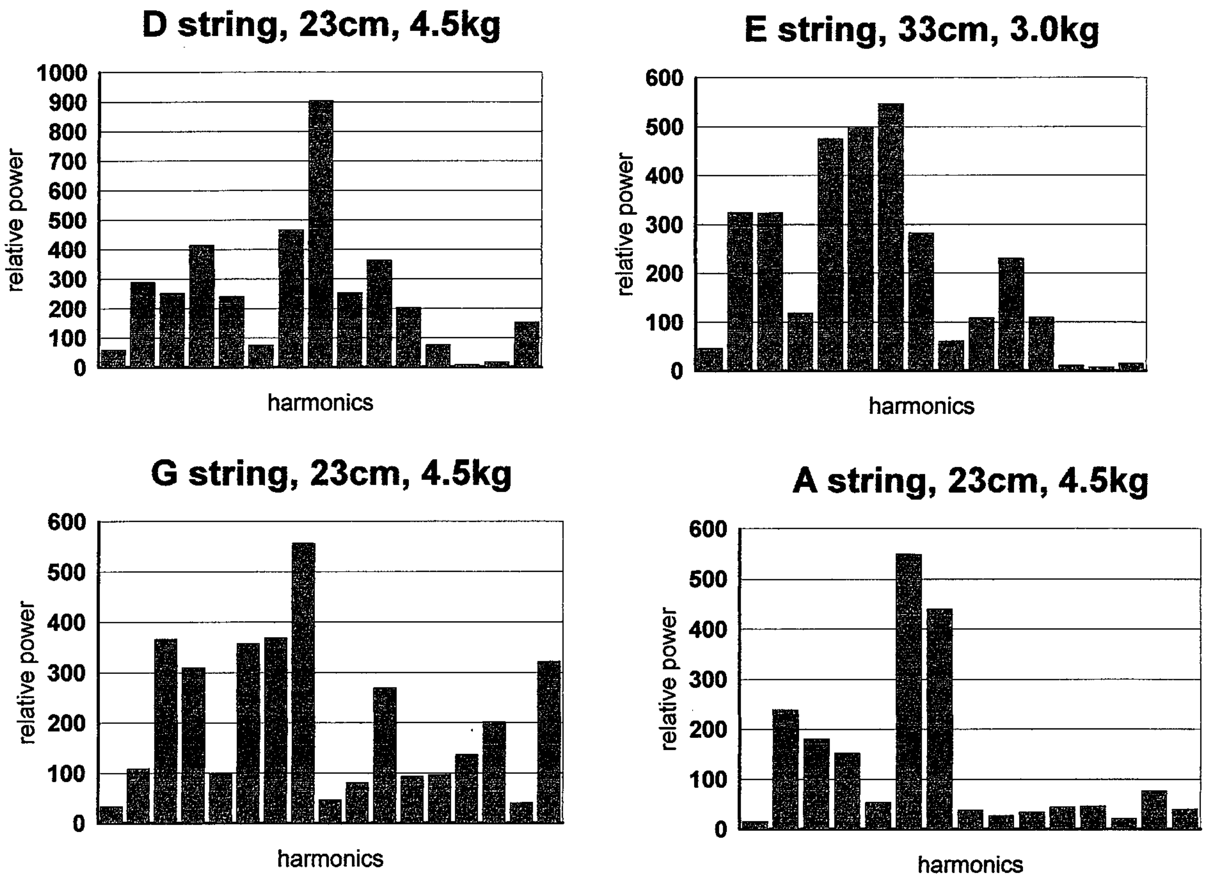

Figure 6: Sample of Relative Harmonics

After looking at these sample plots and the others not included here, several things become apparent. 1) The fundamental is all but missing. 2) The harmonics do not follow any sort of pattern and therefore can not be fit to any function. 3) Right in the middle of the powerful harmonics there is a place where the power drops down very low.

This third observation deserves further study. This drop might be caused by the bow placement. Further measurements were taken to study the affect of bow placement.

Measurements were taken on a violin with the body stuffed to help damp out resonances from the body. Bowing was done at fractional lengths along the string of 1/5, 1/16 and 1/10. The resulting power spectrums are shown here

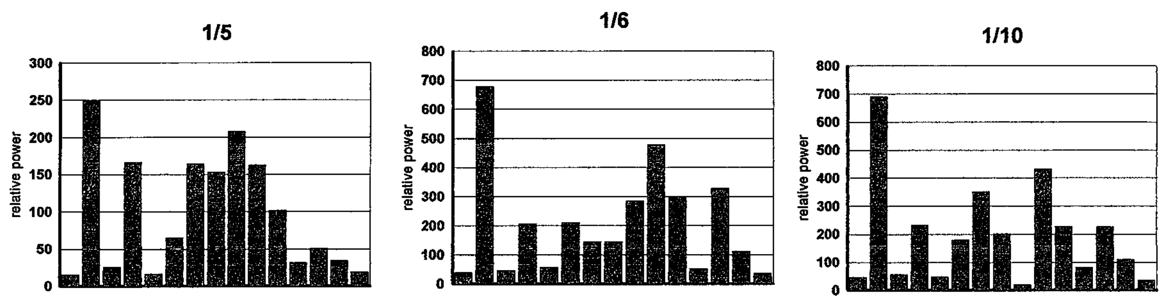

Figure 7: Relative Amplitudes Considering Bow Placement

Considering the finite width of the bow and difficulty of accurate bow placement, these results show that bow placement has a huge affect on the power spectrum. After more research I discovered that bow placement causes ripples in the aforementioned kink on the string, which causes this affect (Fletcher and Rossing 1991).

This presents a problem for the original intent of this project. How can the motion of a bowed string be solved for when something as simple as the bow placement has such a large affect on the power spectrum?

Besides trying to fit a function to the power spectrum, which does not seem possible, it is still possible to find the complex impedance of the bridge and the relation of sound intensity to the impedance of the string and bridge.

As mentioned before, the amount of power that the player can put into the string is proportional to the impedance of the string.

Before dealing with the impedance of the strings, it is necessary to find the frequency of vibration, and then the linear density of the strings.

The pitch was found in each sound measurement from manually counting the number of bytes in one period. The sampling rate of the recording was 44.1k bytes per second, the same sampling rate as a compact disc. The frequency was therefore 44.1k/n, where n is the number of bytes in one period.

Having found the frequency, and knowing the tension and the length, it is now possible to calculate the linear density.

Eq. 14: $\mu = T/(4f^2L^2)$

This is a plot of the calculated linear densities.

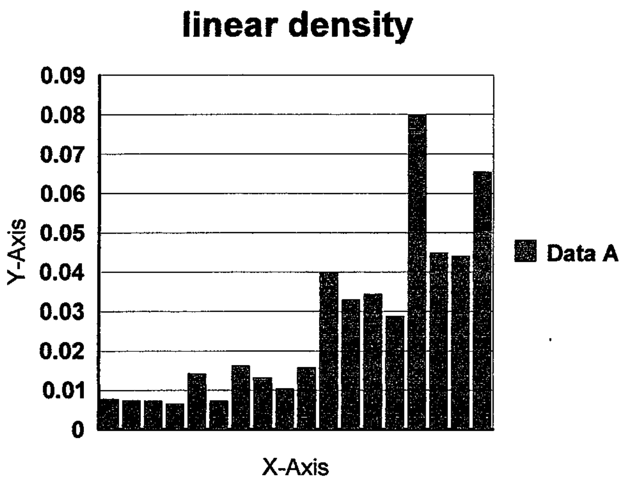

Figure 8: Linear Densities Found by Eq. 14

Units are grams per centimeter.

Every string had four measurements, thus there are four points for each string. The averaged μ for each string is

E: 0.0073 g/cm
A: 0.013 g/cm
D: 0.034 g/cm
G: 0.059 g/cm

From the same measurements above, Impedance (Z) is found by $Z=\sqrt{T\mu}$

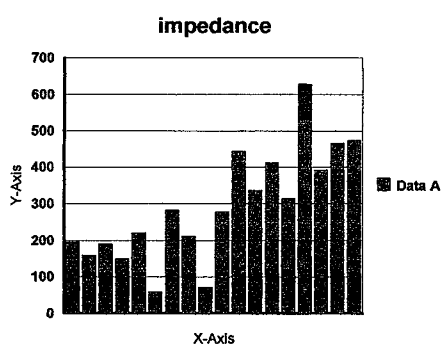

Figure 9: Impedance of Strings

Now we wish to see how impedance and frequency affect intensity. From theory we expect Intensity to be proportional to

$$\frac{Z^2_{string}}{-i\omega Z_{bridge}} \qquad \text{Eq: 15}$$

This is a preliminary plot of Z_{string} vs. Intensity to see if we can spot any relation

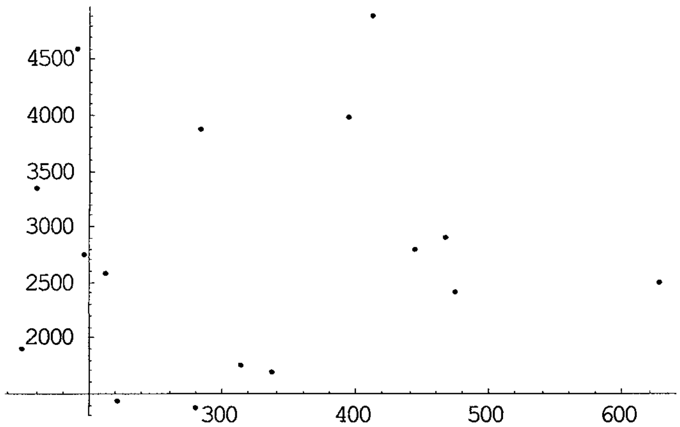

Figure 10: Impedance of String vs. Intensity of Sound

There is a huge amount of noise due to the difficulty in controlling the power input. Even so, an upward trend can be spotted.

From here we want to see if we can find the complex values of Z_{bridge} which is in the form

Eq. 16: $Z = R + K/i\omega$

A plot of ω vs. Z_{bridge} where

Eq. 17: $$Z_{bridge} = \frac{Z_{string}^2}{\omega I}$$

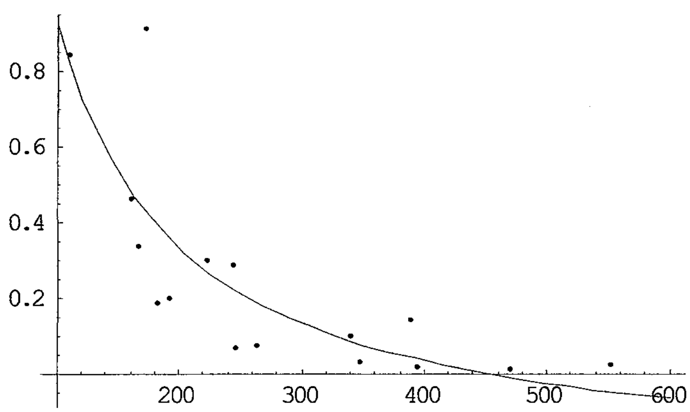

Figure 11: Angular Frequency vs. Impedance of Bridge

The x axis is frequency and the y axis is Z_{bridge}, but the values of Z_{bridge} are arbitrary. The important part is the shape of the curve so that we can find the relation of K to R. The equation of the fit curve is

Eq. 18: $$-0.26362 + \frac{119.178}{x}$$

Eq. 18 is a reiteration of Eq. 13. As was expected, the value of K=119 (due to stiffness of bridge) is much greater than R=0.26 (due to damping).

Discussion

A different method is necessary to attain the goal of this project. After further research it seems like the best method would be to train a neural network. Using a neural network has been used to solve for the complete instrument of several plucked string instruments. A bowed string is much more complicated than a plucked string, but a neural network could solve for just the string.

A neural network works much like an animal brain, developing connections of varying strengths among the neurons. Neural networks are useful in situations where there is a great deal of noise or the function is extremely complicated. Neural networks are being used in voice recognition and signature recognition. Neural networks also need a large number of input and output sets to be trained properly.

This seems to be the best path to solve this problem, but it would still be a very long process. Many more sound samples would be needed. Those sound samples would have to be analyzed and presented to the network as a power spectrum. Presenting the entire sound sample would be far too many data for the network to handle. Accurate measurements of all physical parameters would be necessary. No fitting to find unknown parameters would be possible.

This is the schematic of how the neural network would be used.

Physical Parameters -------> [Neural Network Training] -----> Sound Sample

Sound Sample -------> [Neural Network Testing] -----> Physical Parameters
Physical Parameters -----> [Neural Network] -----> Generated Sound
(to predict how different parameters will affect the sound)

Sound Sample ------> [Neural Network] ------> Physical Parameters
(to judge the parameters of a string by just the sound produced)

This project will require a great deal of work, but at this time I have a much greater understanding of what is required than when I started this project. I see a lot of hope in this endeavor and I would like to continue working toward the original goal.

Bibliography:

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Morse, Philip M. and Ingard, K. Uno; *Theoretical Acoustics,* Princeton University Press, 1968