Homework #1

1.2) i) $\bar{x} = \frac{\sum x_i}{N}$ $\quad N = 13$

$$\bar{x} = \frac{0+1+2+\cdots+12}{13} = \frac{78}{13} = 6$$

$$s^2 = \frac{1}{N-1} \sum (x_i - \bar{x})^2$$

$$= \frac{1}{12}\left[2\times 6^2 + 2\times 5^2 + 2\times 4^2 + 2\times 3^2 + 2\times 2^2 + 1\right]$$

$$= \frac{182}{12} = 15.17 \qquad s = 3.89$$

$$u^2 = \frac{s^2}{N} = \frac{15.17}{13} = 1.17 \qquad u = 1.08 \simeq 1.1$$

ii) $$\bar{x} = \frac{\sum m_i x_i}{\sum m_i} = \frac{(0+1\times1+2\times2+3\times3+4\times4+5\times5+6\times6 + 7\times5+8\times4+9\times3+10\times2+11\times1+0)}{0+1+2+3+\cdots+2+1+0}$$

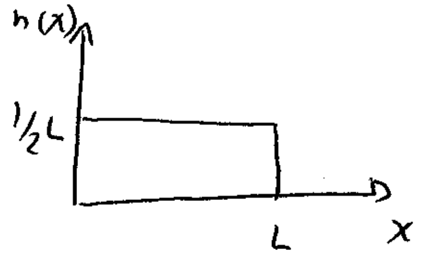
$$= \frac{216}{36} = 6.00$$

$$s^2 = \frac{\sum m_i (x_i - \bar{x})^2}{\sum m_i - 1} = \frac{(0+1\times5^2+2\times4^2+3\times3^2+4\times2^2+5+0 + 5+4\times2^2+3\times3^2+2\times4^2+5^2+0)}{36-1}$$

$$= \frac{210}{35} = 6.00 \qquad s = 2.45$$

$$u^2 = \frac{s^2}{\sum m_i} = \frac{6}{36} = 0.167 \qquad u = 0.408 \simeq 0.41$$

(iii)



$$n(x) = \begin{cases} \frac{1}{L} & 0 \le x \le L \\ 0 & \text{otherwise} \end{cases}$$

$$\int_0^L n(x)\,dx = \frac{1}{L}\int_0^L dx = \frac{x}{L}\Big|_0^L = 1$$

$$\bar{x} = \int_{\text{all}_x} x\, n(x)\,dx = \frac{1}{L}\int_0^L x\,dx = \frac{1}{2}\frac{1}{L}x^2\Big|_0^L = \frac{L}{2}$$

obvious from figure

$$S^2 = \int_{\text{all}_x} (x-\bar{x})^2 n(x)\,dx = \frac{1}{L}\int_0^L (x - L/2)^2\,dx$$

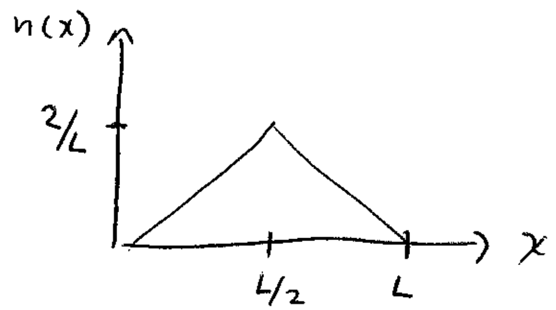
$$= \frac{1}{L}\int_0^L x^2\,dx + \frac{L}{2}\int_0^L dx - \int_0^L x\,dx$$

$$= \frac{1}{3}L^2 + \frac{1}{4}L^2 - \frac{1}{2}L^2 = \frac{L^2}{12}$$

$$S = \frac{L}{2\sqrt{3}} = 0.29\,L$$

(iv)

$$n(x) = \begin{cases} \frac{4x}{L^2} & \text{for } 0 \le x \le L/2 \\ \frac{4(L-x)}{L^2} & \text{for } \frac{L}{2} \le x \le L \end{cases}$$



from graph, it is clear that $\bar{x} = L/2$ and area is 1

<u>Always</u> <u>make</u> <u>a</u> <u>graph</u>

$$\bar{x} = \int_{\text{all}_x} x\, n(x)\,dx = \int_0^{L/2} \frac{4x^2}{L^2}\,dx + \int_{L/2}^{L} \frac{4(L-x)x}{L^2}\,dx$$

$$= \frac{4}{L^2}\int_0^{L/2} x^2\,dx + \frac{4}{L}\int_{L/2}^{L} x\,dx - \frac{4}{L^2}\int_{L/2}^{L} x^2$$

$$= L/6 + 2L - L/2 - \frac{4L}{3} + L/6 = L/2 \quad \text{As expected}$$

Homework #1

1.3)

i) $P(t)\,dt = \frac{1}{\tau} e^{-t/\tau}\,dt$ for $t \geq 0$

$$\int_0^\infty P(t)\,dt = -e^{-t/\tau}\Big|_0^\infty = -0+1 = 1$$ properly normalized

ii) $$\bar{t} = \int_0^\infty t\,P(t)\,dt = \frac{1}{\tau}\int_0^\infty t\,e^{-t/\tau}\,dt$$

$$= \frac{1}{\tau}\left[\tau\int_0^\infty e^{-t/\tau}dt + te^{-t/\tau}\Big|_0^\infty\right] = \int_0^\infty e^{-t/\tau}dt$$

$$= -\tau e^{-t/\tau}\Big|_0^\infty = \tau \quad \Rightarrow \bar{t} = \tau$$

iii) $$\overline{(t-\tau)^2} = \frac{1}{\tau}\int_0^\infty (t-\tau)^2 e^{-t/\tau}\,dt$$

$$= \frac{1}{\tau}\int_0^\infty (t^2 - 2t\tau + \tau^2)\,e^{-t/\tau}\,dt$$

$$= \frac{1}{\tau}\left[\int_0^\infty t^2 e^{-t/\tau}dt - 2\tau^3 + \tau^2\int_0^\infty e^{-t/\tau}dt\right]$$

$$= \frac{1}{\tau}\int_0^\infty t^2 e^{-t/\tau}dt - \tau^2$$

$$= \frac{2}{\tau}\int_0^\infty t\,\tau\,e^{-t/\tau}\,dt - \tau^2 = 2\tau^2 - \tau^2 = \tau^2$$

iv) $$\bar{t} = \frac{2.75}{10} = 0.275$$ in units of 10^{-12} s

$$u^2 = \frac{S^2}{N} = \frac{\overline{(t-\tau)^2}}{10} \Rightarrow \frac{(0.275)^2}{10} \qquad u = 0.17$$

or

$$u^2 = \frac{1}{(N-1)N}\Sigma(x_i - \bar{x})^2 \qquad u = 0.09$$

$$s^2 = \int_{\text{all } x} (x-\bar{x})^2 n(x)dx = \int_0^{L/2} \frac{4x}{L^2}(x-L/2)^2 dx + \int_{L/2}^{L} \frac{4(L-x)}{L^2}(x-L/2)^2 dx$$

these are same, in 2nd term set $y = L-x$ to see

$$= \frac{8}{L^2}\int_0^{L/2} x(x^2 + L^2/4 - xL)dx$$

$$= \frac{8}{L^2}\int_0^{L/2} x^3 dx + 2\int_0^{L/2} x\,dx - \frac{8}{L}\int_0^{L/2} x^2 dx$$

$$= \left(\frac{1}{8} + \frac{1}{4} - \frac{1}{3}\right)L^2 = 0.0417 L^2$$

$$S = 0.204L \simeq 0.20L$$

v) for

$$\left.\begin{array}{ll} \text{i)} & \frac{s}{\bar{x}} = 0.65 \\ \text{iii)} & \frac{s}{\bar{x}} = 0.58 \end{array}\right\}$$ similar because flat distributions

$$\left.\begin{array}{ll} \text{ii)} & \frac{s}{\bar{x}} = 0.41 \\ \text{iv)} & \frac{s}{\bar{x}} = 0.41 \end{array}\right\}$$ similar, trangular distributions

Homework #1

1.4) 165.6 ± 0.3
165.1 ± 0.4
166.4 ± 1.0
166.1 ± 0.8

$$\bar{x} = \frac{\sum \frac{x_i}{\sigma_i^2}}{\sum \frac{1}{\sigma_i^2}} \qquad \frac{1}{\sigma^2} = \sum \frac{1}{\sigma_i^2}$$

$$\bar{x} = \frac{\frac{165.6}{(0.3)^2} + \frac{165.1}{(0.4)^2} + \frac{166.4}{1} + \frac{166.1}{(0.8)^2}}{\frac{1}{(0.3)^2} + \frac{1}{(0.4)^2} + 1 + \frac{1}{(0.8)^2}} = \frac{3298}{19.92}$$

$$\bar{x} = 165.5 \pm 0.2 \qquad \sigma = \frac{1}{\sqrt{19.92}} = 0.22$$

second part, using just the numbers, yields
$\bar{x} = 165.8 \pm 0.3$

1.9) $l \pm \sigma_l$, $b + \sigma_b$

$$A = l\,b \qquad \sigma_A^2 = \left(\frac{\partial A}{\partial l}\right)^2 \sigma_l^2 + \left(\frac{\partial A}{\partial b}\right)^2 \sigma_b^2$$

$$\sigma_A^2 = b^2\sigma_l^2 + l^2\sigma_b^2$$

$$\sigma_A = \sqrt{b^2\sigma_l^2 + l^2\sigma_b^2} = \sqrt{\frac{\sigma_l^2}{l^2} + \frac{\sigma_b^2}{b^2}}$$

$$P = l + b$$

$$\sigma_P^2 = \sigma_l^2 + \sigma_b^2$$

$$\sigma_P = \sqrt{\sigma_l^2 + \sigma_b^2}$$

σ_A + σ_P both depend on σ_l + σ_b and so they are correlated

2.1) a) $a_1 \pm \sigma_1 \quad , \quad a_2 \pm \sigma_2$

$$S = \left(\frac{a_1 - \bar{a}}{\sigma_1}\right)^2 + \left(\frac{a_2 - \bar{a}}{\sigma_2}\right)^2$$

to optimize

$$\frac{\partial S}{\partial \bar{a}} = -\frac{2(a_1 - \bar{a})}{\sigma_1^2} - \frac{2(a_2 - \bar{a})}{\sigma_2^2} = 0$$

$$\frac{\bar{a} - a_1}{\sigma_1^2} + \frac{\bar{a} - a_2}{\sigma_2^2} = 0$$

solve for $\bar{a}$

$$\bar{a} = \frac{\frac{a_1}{\sigma_1^2} + \frac{a_2}{\sigma_2^2}}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}$$

$$\sigma^2 = \left(\frac{1}{2}\frac{\partial^2 S}{\partial \bar{a}^2}\right)^{-1} \Rightarrow \frac{1}{\sigma^2} = \frac{1}{2}\frac{\partial^2 S}{\partial \bar{a}^2}$$

from above $\displaystyle \frac{\partial^2 S}{\partial \bar{a}^2} = 2\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)$

$$\frac{1}{\sigma^2} = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}$$

b) $L = \alpha a_1 + (1-\alpha) a_2 \qquad \sigma^2 = \left(\frac{\partial L}{\partial a_1}\right)^2 \sigma_1^2 + \left(\frac{\partial L}{\partial a_2}\right)^2 \sigma_2^2$

$$\frac{\partial \sigma^2}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left[\alpha^2 \sigma_1^2 + (1-\alpha)^2 \sigma_2^2\right] = 0 \quad \text{for minimum}$$

$$\Rightarrow \alpha \sigma_1^2 + (\alpha - 1)\sigma_2^2 = 0 \Rightarrow \alpha = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

$$L = \frac{\sigma_2^2 a_1}{\sigma_1^2 + \sigma_2^2} + \frac{\sigma_1^2 a_2}{\sigma_1^2 + \sigma_2^2} = \frac{\frac{a_1}{\sigma_1^2} + \frac{a_2}{\sigma_2^2}}{1/\sigma_1^2 + \frac{1}{\sigma_2^2}} = \bar{a}$$

$$\sigma_{min}^2 = \alpha^2 \sigma_1^2 + (1-\alpha)^2 \sigma_2^2$$

$$= \frac{\sigma_2^4 \sigma_1^2}{(\sigma_1^2 + \sigma_2^2)^2} + \frac{\sigma_1^4 \sigma_2^2}{(\sigma_1^2 + \sigma_2^2)^2} = \frac{(\sigma_2^2 + \sigma_1^2)\sigma_1^2 \sigma_2^2}{(\sigma_2^2 + \sigma_1^2)^2}$$

$$\sigma_{min}^2 = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} = \frac{1}{1/\sigma_1^2 + 1/\sigma_2^2}$$

so everything is consistent

2. 2)

$$m_1 = (0.9 \pm 0.1) \quad \text{sol mass}$$
$$m_2 = (1.4 \pm 0.2) \quad \text{sol mass}$$

$$\chi^2 = \frac{(m_1 - \bar{m})^2}{\sigma_1^2} + \frac{(m_2 - \bar{m})^2}{\sigma_2^2}$$

$$\bar{m} = \frac{\frac{m_1}{\sigma_1^2} + \frac{m_2}{\sigma_2^2}}{1/\sigma_1^2 + \frac{1}{\sigma_2^2}} = 1.0 \quad \text{sol mass}$$

$$\chi^2 = \frac{(0.1)^2}{(0.1)^2} + \frac{(0.4)^2}{(0.2)^2} = 5.0$$

one degree of freedom
from Fig. 2.7, this corresponds to ~ 3%
(or table A6.3)

A possible test would be to use

$$f_i = \frac{|m_i - \bar{m}|}{\sigma_i}$$

for m_1 $\quad f_1 = \frac{|0.9 - 1.0|}{0.1} = 1.0$

m_2 $\quad f_2 = \frac{|1.4 - 1.0|}{0.2} = 2.0$

for $r > 1.0$ ~ 25% } probability of
$r > 2.0$ ~ 3% } being consistent

2.5)

$$N = N_0 e^{-t/\tau}$$

$$\delta N_i = \sqrt{N_i}$$

$$\ln N = \ln N_0 - t/\tau$$

$$\delta^2(\ln N) = \left(\frac{\partial \ln N}{\partial N}\right)^2 \delta^2 N_i = \left(\frac{1}{N}\right)^2 N = \frac{1}{N}$$

$$\delta(\ln N) = \frac{1}{\sqrt{N}}$$

let $y = \ln N \qquad \delta y = \frac{1}{\sqrt{N}}$

$$y = y_0 - t/\tau$$

$$\tau = (21.7 \pm 0.2)\,\mu s$$

2.3) i) $\sum \frac{a+bx_i-y_i}{\sigma_i^2} = 0$ $\qquad$ $\sum \frac{(a+bx_i-y_i)x_i}{\sigma_i^2} = 0$

$$a = \frac{b\sum \frac{x_i}{\sigma_i^2} - \sum \frac{y_i}{\sigma_i^2}}{\sum 1/a_i^2} \qquad a = \frac{b\sum \frac{x_i^2}{\sigma_i^2} - \sum \frac{x_i y_i}{\sigma_i^2}}{\sum \frac{x_i}{\sigma_i^2}}$$

$$b\left(\sum \frac{x_i}{\sigma_i^2}\right)\left(\sum \frac{x_i}{\sigma_i^2}\right) - \left(\sum \frac{x_i}{\sigma_i^2}\right)\left(\sum \frac{y_i}{\sigma_i^2}\right)$$

$$= b\left(\sum \frac{1}{\sigma_i^2}\right)\left(\sum \frac{x_i^2}{\sigma_i^2}\right) - \left(\sum \frac{1}{\sigma_i^2}\right)\left(\sum \frac{x_i y_i}{\sigma_i^2}\right)$$

$$b = \frac{\left(\sum \frac{1}{\sigma_i^2}\right)\left(\sum \frac{x_i y_i}{\sigma_i^2}\right) - \left(\sum \frac{x_i}{\sigma_i^2}\right)\left(\sum \frac{y_i}{\sigma_i^2}\right)}{\left(\sum \frac{1}{\sigma_i^2}\right)\left(\sum \frac{x_i^2}{\sigma_i^2}\right) - \left(\sum \frac{x_i}{\sigma_i^2}\right)^2}$$

$$b = \frac{[1][xy] - [x][y]}{[1][x^2] - [x]^2}$$

ii) $\frac{1}{\sigma^2} = \sum \frac{1}{\sigma_i^2} = [1]$

if $\sigma_i = \sigma'$ for all i

$$\frac{1}{\sigma^2} = \frac{1}{\sigma'^2}\sum = \frac{n}{\sigma'^2}$$

$$\sigma = \frac{\sigma'}{\sqrt{n}}$$